Semantic Theory
Lecture 2: First-Order predicate Logic

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Predicate Logic – Vocabulary

- **Non-logical expressions:**
  - Individual constants: CON
  - n-place relation constants: $PRED^n$, for all $n \geq 0$
- **Infinite set of individual variables:** VAR
- **Logical connectives:** ¬, ∧, ∨, →, ↔
- **Quantifiers:** ∀, ∃
- **Identity relation:** =
- **Brackets:** (,)

Predicate Logic – Syntax

- **Terms:** TERM = VAR ∪ CON
- **Atomic formulas:**
  - $R(t_1, \ldots, t_n)$ for $R \in PRED^n$ and $t_1, \ldots, t_n \in TERM$
  - $t_1 = t_2$ for $t_1, t_2 \in TERM$
- **Well-formed formulas:** the smallest set WFF such that
  - all atomic formulas are WFF
  - if φ and ψ are WFF, then $\neg\varphi$, $(\varphi \land \psi)$, $(\varphi \lor \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$ are WFF
  - if $x \in VAR$, and φ is a WFF, then $\forall x\varphi$ and $\exists x\varphi$ are WFF

Free and Bound Variables

- If ∀xφ (∃xφ) is a subformula of a formula ψ, then φ is the **scope** of this occurrence of ∀x (∃x) in ψ.
- An *occurrence* of variable x in a formula φ is **free in φ** if this occurrence of x does not fall within the scope of a quantifier ∀x or ∃x in φ.
- If ∀xψ (∃xψ) is a subformula of φ and x is free in ψ, then this occurrence of x is **bound by** this occurrence of the quantifier ∀x (∃x).
- A **closed formula** is a formula without free variables.

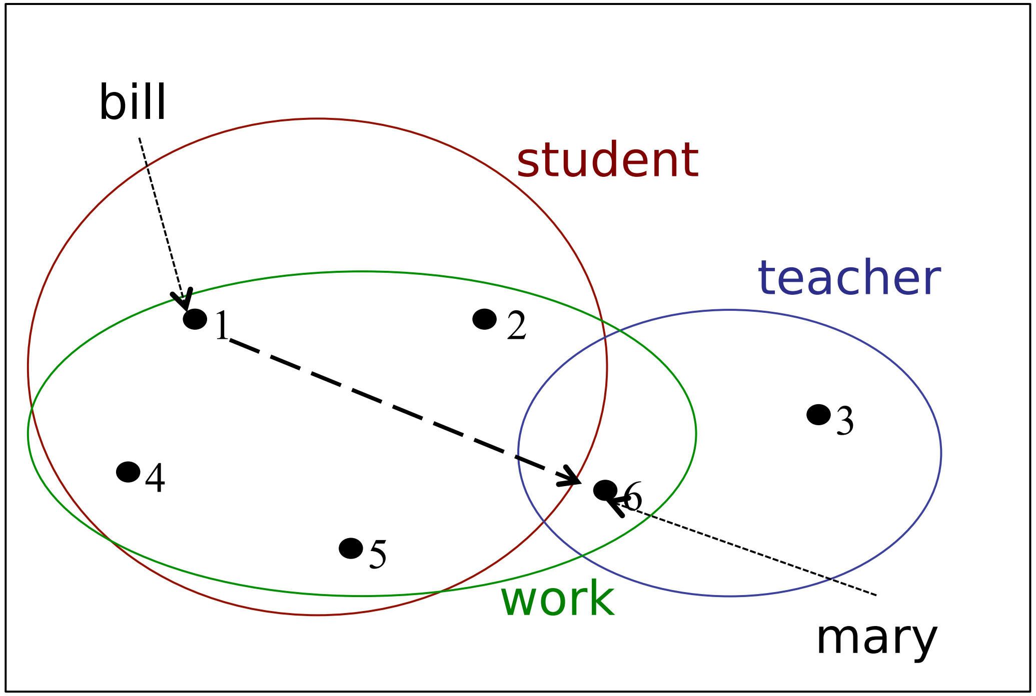
Predicate Logic – Semantics

- Expressions of Predicate Logic are interpreted relative to **model structures** and **variable assignments**.
- Model structures are our “mathematical picture” of the world: They provide interpretations for non-logical symbols (predicate symbols, individual constants).
- Variable assignments provide interpretations for variables.

Model structures

- **Model structure:** $M = \langle U_M, V_M \rangle$
  - U_M is non-empty set – the “universe”
  - V_M is an interpretation function assigning individuals ($\in U_M$) to individual constants and n-ary relations over U_M to n-place predicate symbols:
    - $V_M(P) \in \{0,1\}$ if P is an 0-place predicate symbol
    - $V_M(P) \subseteq U_M^n$ if P is an n-place predicate symbol
    - $V_M(c) \in U_M$ if c is an individual constant
- **Assignment function** for variables $g: VAR \rightarrow U_M$

Model Structure, Example



$M = \langle U_M, V_M \rangle$

$U_M = \{ 1, 2, 3, 4, 5, 6\}$

$V_M(\text{bill}) = 1$

$V_M(\text{mary}) = 5$

$V_M(\text{student}) = \{ 1, 2, 4, 5\}$

$V_M(\text{teacher}) = \{ 3, 6 \}$

$V_M(\text{work}) = \{ 1, 2, 4, 5, 6\}$

$V_M(\text{like}) = \{ \langle 1, 6 \rangle \}$

Interpretation of Terms

- **Interpretation of terms** with respect to a model structure M and a variable assignment g:

$$[\![\alpha]\!]^{M,g} = \begin{cases} V_M(\alpha) & \text{if } \alpha \text{ is an individual constant} \\ g(\alpha) & \text{if } \alpha \text{ is a variable} \end{cases}$$

Interpretation of Formulas

- **Interpretation of formulas** with respect to a model structure M and variable assignment g:
  - $[\![R(t_1, \ldots, t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$
  - $[\![t_1 = t_2]\!]^{M,g} = 1$ iff $[\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$
  - $[\![\neg\varphi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$
  - $[\![\varphi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
  - $[\![\varphi \vee \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$
  - $[\![\varphi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$
  - $[\![\varphi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = [\![\psi]\!]^{M,g}$
  - $[\![\exists x\varphi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\varphi]\!]^{M,g[x/d]} = 1$
  - $[\![\forall x\varphi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\varphi]\!]^{M,g[x/d]} = 1$

Variable assignments

- We write **g[x/d]** for the assignment that assigns d to x and assigns the same values as g to all other variables.
  - g[x/d](y) = d, if x = y
  - g[x/d](y) = g(y), if x ≠ y

	x	**y**	**z**	**u**	**…**
g	a	b	c	d	…
g[x/a]	a	b	c	d	…
g[y/a]	a	a	c	d	…
g[y/g(z)]	a	c	c	d	…
g[y/a][u/a]	a	a	c	a	…
g[y/a][y/b]	a	b	c	d	…

Truth, Validity, Entailment

- **A formula φ is true in a model structure M** iff $[\![\varphi]\!]^{M,g} = 1$ for every variable assignment g.
- **A formula φ is valid (⊨ φ)** iff φ is true in all model structures.
- **A formula φ is satisfiable** iff there is at least one model structure M such that φ is true in M.
- **A set of formulas Γ is (simultaneously) satisfiable** iff there is a model structure M such that every formula in Γ is true in M (“M satisfies Γ,” or “M is a model of Γ”).
- **Γ entails a formula φ (Γ ⊨ φ)** iff φ is true in every model structure that satisfies Γ.

Logical Equivalence

- **Formula φ is logically equivalent to formula ψ (φ⇔ψ)**, iff

  $[\![\varphi]\!]^{M,g} = [\![\psi]\!]^{M,g}$ for all model structures M and variable assignments g.

- For all **closed formulas** φ and ψ, the following assertions are equivalent:

  1. **φ ⊨ ψ and ψ ⊨ φ** (mutual entailment)
  2. **φ⇔ψ** (logical equivalence)
  3. ⊨ **φ** ↔ **ψ** (validity of “material equivalence”)

- Problem: Why does this hold for closed formulas only? What is the situation in the case of formulas containing free variables (like “Fx” and “Fy”)?

The Principle of Extensionality

- Theorem: Let φ be a subformula of χ, [ψ/φ]χ be the result of replacing φ in χ with ψ:

  If $\varphi \Leftrightarrow \psi$, then $\chi \Leftrightarrow [\psi/\varphi]\chi$

- The theorem states the theoretically important **Principle of Extensionality**.
- An important practical consequence of the theorem is that it justifies equivalence transformations of logical formulas by substituting sub-expressions with logically equivalent ones.

Some Useful Logical Theorems Involving Connectives

1)	$\neg\neg\varphi \Leftrightarrow \varphi$	Double negation
2)	$\varphi\wedge\psi \Leftrightarrow \psi\wedge\varphi$	Commutativity of ∧, ∨
3)	$\varphi\vee\psi \Leftrightarrow \psi\vee\varphi$	
4)	$\varphi\wedge(\psi\vee\chi) \Leftrightarrow (\varphi\wedge\psi)\vee(\varphi\wedge\chi)$	Distributivity of ∧ and ∨
5)	$\varphi\vee(\psi\wedge\chi) \Leftrightarrow (\varphi\vee\psi)\wedge(\varphi\vee\chi)$	
6)	$\neg(\varphi\wedge\psi) \Leftrightarrow \neg\varphi\vee\neg\psi$	de Morgan’s Law
7)	$\neg(\varphi\vee\psi) \Leftrightarrow \neg\varphi\wedge\neg\psi$	
8)	$\varphi \rightarrow\neg\psi \Leftrightarrow \psi \rightarrow\neg\varphi$	Law of Contraposition
9)	$\varphi \rightarrow \psi \Leftrightarrow \neg\varphi\vee\psi$	→ and ∨
10)	$\neg(\varphi \rightarrow \psi) \Leftrightarrow \varphi\wedge\neg\psi$	→ and ∧

Some Useful Logical Theorems Involving Quantifiers

11) $\neg\forall x\varphi \Leftrightarrow \exists x\neg\varphi$ Quantifier negation

12) $\neg\exists x\varphi \Leftrightarrow \forall x\neg\varphi$

13) $\forall x(\varphi \wedge \Psi) \Leftrightarrow \forall x\varphi \wedge \forall x\Psi$ Quantifier distribution

14) $\exists x(\varphi \vee \Psi) \Leftrightarrow \exists x\varphi \vee \exists x\Psi$

15) $\forall x\forall y\varphi \Leftrightarrow \forall y\forall x\varphi$ Quantifier Swap

16) $\exists x\exists y\varphi \Leftrightarrow \exists y\exists x\varphi$

17) $\exists x\forall y\varphi \Rightarrow \forall y\exists x\varphi$ **...** but not vice versa !

Provided that y does not occur free in φ, the following holds:

18) $\exists x\varphi \Leftrightarrow \exists y\varphi[y/x]$

19) $\forall x\varphi \Leftrightarrow \exists y\varphi[y/x]$, where $\varphi[y/x]$ is the result of replacing all free occurrences of x (i.e.: all occurrences of x bound by the quantifier) with y.

Some More Theorems: Quantifier Shift

The following equivalences are valid theorems of FOL, provided that x does not occur free in φ:

20) $\varphi \wedge \forall x \Psi \Leftrightarrow \forall x(\varphi \wedge \Psi)$
21) $\varphi \wedge \exists x \Psi \Leftrightarrow \exists x(\varphi \wedge \Psi)$
22) $\varphi \vee \forall x \Psi \Leftrightarrow \forall x(\varphi \vee \Psi)$
23) $\varphi \vee \exists x \Psi \Leftrightarrow \exists x(\varphi \vee \Psi)$

24) $\varphi \rightarrow \forall x \Psi \Leftrightarrow \forall x(\varphi \rightarrow \Psi)$
25) $\varphi \rightarrow \exists x \Psi \Leftrightarrow \exists x(\varphi \rightarrow \Psi)$
26) $\exists x \Psi \rightarrow \varphi \Leftrightarrow \forall x(\Psi \rightarrow \varphi)$
27) $\forall x \Psi \rightarrow \varphi \Leftrightarrow \exists x(\Psi \rightarrow \varphi)$

Equivalence Transformations: An Example

(1) $\neg\exists x \forall y (Py \rightarrow Rxy)$ ("*Nobody masters every problem*")

(2) $\forall x \exists y (Py \land \neg Rxy)$ (*"Everybody fails to master some problem"*)

We show the equivalence of (1) and (2) as follows:

$\neg\exists x \forall y (Py \rightarrow Rxy) \Leftrightarrow \forall x \neg\forall y (Py \rightarrow Rxy)$ $(\neg\exists x \varphi \Leftrightarrow \forall x \neg\varphi)$

$\forall x \underline{\neg\forall y (Py \rightarrow Rxy)} \Leftrightarrow \forall x \underline{\exists y \neg(Py \rightarrow Rxy)}$ $(\neg\exists x \varphi \Leftrightarrow \forall x \neg\varphi)$

$\forall x \exists y \underline{\neg(Py \rightarrow Rxy)} \Leftrightarrow \forall x \exists y (\underline{Py \land \neg Rxy})$ $(\neg(\varphi \rightarrow \psi) \Leftrightarrow \varphi \land \neg\psi)$